Multiobjective Programming

Minghe Sun
College of Business, University of Texas at San Antonio, USA

INTRODUCTION

Many optimization problems have multiple criteria or objectives. Examples of multiobjective optimization problems are in all areas of businesses. In a financial or capital investment portfolio selection problem, the objectives may be maximization of portfolio return, minimization of risk, maximization of dividend income, maximization of amount invested in R&D, maximization of social responsibility, maximization of liquidity, *etc*. (Steuer, Qi, & Hirschberger et al., 2005; Stummer & Sun, 2005). In a supply chain network problem, the objectives may be fair profit distribution among all participants, safe inventory levels, maximum customer service levels, and robustness of decision to uncertain product demands (Chen & Lee, 2004). In a production scheduling problem, the objectives to be considered may include the minimization of the makespan, minimization of the total flow time, and the minimization of machine idleness (Lei, 2009). In transportation and distribution problems, the objectives may include the minimization of the total cost incurred, the minimization of the total time taken to deliver the commodities, and the minimization of the total distances traveled (Sun, 2003). In an information flow network, the objectives may include the maximization of the volume of data flow, the minimization of the time the data take to travel, and the minimization of the cost of transferring the data (Sun, 2003). Multiobjective optimization problems can be represented and solved through multiobjective programming models. A multiobjective programming model is a mathematical programming model with more than one objective function.

Theory and solution techniques for multiobjective programming problems have been developed in the last 50 years. The theoretical development greatly helped the understanding and guided the development of solution techniques in multiobjective programming. Because there is not a unique solution that optimizes all the objective functions simultaneously in a general multiobjective programming problem, preference information is needed from the decision maker (DM) in the solution process to judge the quality of the trial solutions in order to make tradeoffs. Solution procedures are then usually classified based on the stage of the solution process in which the preference information is needed from the DM (Hwang & Masud, 1979). These approaches are (a) those requiring no preference information, (b) those requiring *a priori* articulation of the DM's preference information, (c) those requiring progressive articulation of the DM's preference information, and (d) those requiring *a posteriori* articulation of the DM's preference information.

In this chapter, the model, concepts and definitions of multiobjective programming are explained. Different solution methods are also discussed. Examples are provided to help in presenting the material and graphics are used in these examples whenever convenient.

THE MULTIOBJECTIVE PROGRAMMING MODEL AND RELATED CONCEPTS

Without loss of generality, all objective functions are maximized. A multiobjective programming model can be written as

$$\max\ z_k = f_k(\mathbf{x}) \text{ for } 1 \le k \le p$$

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s.t. $g_i(\mathbf{x}) \le b_i$ for $1 \le i \le m$

$\mathbf{l} \le \mathbf{x} \le \mathbf{u}$

In the model, $\mathbf{x} \in \Re^n$ is a vector of decision variables, $f_k(\mathbf{x})$ is the kth objective function, $g_i(\mathbf{x})$ is the ith constraint, and $\mathbf{l} \in \Re^n$ and $\mathbf{u} \in \Re^n$ are the lower and upper bounds on $\mathbf{x}$.

Most of the following concepts are from Steuer (1986). The set $X = \{\mathbf{x} \in \Re^n \mid g_i(\mathbf{x}) \le b_i$, for $1 \le i \le m$ and $\mathbf{l} \le \mathbf{x} \le \mathbf{u}\}$ is the feasible region in decision space. A point $\mathbf{x} \in X$ is a feasible solution in decision space. The set

$Z = \{\mathbf{z} \in \Re^p \mid z_k = f_k(\mathbf{x})$ for all $1 \le k \le p$,

$\mathbf{x} \in X\}$ is the feasible region in criterion space. A $\mathbf{z} \in Z$ is a feasible solution in criterion space or a criterion vector. A criterion vector $\bar{\mathbf{z}} \in Z$ is nondominated if there does not exist another criterion vector $\mathbf{z} \in Z$, such that $\mathbf{z} \ge \bar{\mathbf{z}}$ and $\mathbf{z} \ne \bar{\mathbf{z}}$. $\bar{Z}$ is used to represent the set of all nondominated solutions. A $\bar{\mathbf{x}} \in X$ is efficient if $\bar{\mathbf{z}} \in \bar{Z}$, such that $\bar{z}_k = f_k(\bar{\mathbf{x}})$ for all $1 \le k \le p$. $\bar{X}$ is used to represent the set of all efficient solutions. Nondominated or efficient solutions are sometimes called Pareto optimal solutions. A criterion vector $\hat{\mathbf{z}} \in Z$ is optimal if it maximizes the DM's value function. However, a DM's value function in real life problems is hard to estimate or not assessable and its functional form is usually unavailable (Yu, 1985). If $\hat{\mathbf{z}}$ is optimal, $\hat{\mathbf{z}} \in \bar{Z}$, *i.e.*, an optimal solution must be nondominated. A point

$\mathbf{z}^* \in \Re^k$,

such that

$z_k^* = \max\{f_k(\mathbf{x}) \mid \mathbf{x} \in X\}$

for all $1 \le k \le p$, is the ideal solution or ideal point. For most problems,

$\mathbf{z}^* \notin Z$.

A point

$\mathbf{x}^* \in \Re^n$,

such that

$z_k^* = f_k(\mathbf{x}^*)$ f

or all $1 \le k \le p$, does not usually exist (Sun, 2005). A point

$\mathbf{z}^{**} \in \Re^k$,

such that

$z_k^{**} = z_k^* + \varepsilon_k$,

where $\varepsilon_k > 0$ and small, is a utopian point.

Nondominated solutions may be supported or unsupported. The set $Z^{\ge}$ is defined as the convex hull of $\{\bar{Z} \oplus \{\mathbf{z} \in \Re^K \mid \mathbf{z} \ge 0\}\}$ (Steuer, 1986), where $\oplus$ signifies vector addition. A criterion vector $\mathbf{z} \in \bar{Z}$ is supported if it is on the boundary of $Z^{\ge}$ and is unsupported otherwise. A point $\mathbf{x} \in \bar{X}$ is a supported efficient solution if $\mathbf{z}$ is supported and is an unsupported efficient solution if $\mathbf{z}$ is unsupported such that $z_k = f_k(\mathbf{x})$ for all $1 \le k \le p$. All nondominated solutions are supported in continuous linear problems. Nonlinear and integer problems may have unsupported nondominated solutions. Unsupported nondominated solutions cannot be reached by a supporting hyper plane and cannot be found by optimizing a weighted sum of linear objective functions. Because unsupported nondominated solutions are also candidates for an optimal solution, special techniques are needed to find them.