Nonlinear Programming

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INTRODUCTION

Nonlinear programming (NLP) deals with the problem of optimizing an objective function in the presence of equality and inequality constraints, where some of the constraints or the objective functions are nonlinear. A general optimization problem is to select n decision variables $x_1, x_2, \ldots x_n$ from a given feasible region in such a way as to optimize (minimize or maximize) a given objective function $f(x_1, x_2, \ldots x_n)$ of the decision variables. The problem is called a nonlinear programming problem (NLP) if the objective function is nonlinear and/or the feasible region is determined by nonlinear constraints.

For notational convenience, we usually let x denote the vector of n decision variables $x_1, x_2, \ldots x_n$ that is, $x = (x_1, x_2, \ldots x_n)$ and write the problem more concisely. Throughout the article, we shall consider the following NLP problem:

Minimize *f (x),*

Subject to

$g(x) \geq 0$

$h(x) = 0$

$x \in X$

where X is a subset of R^{n_x}, x is a vector of n_x components $x_1, \ldots x_{n_x}$, and f: X $\rightarrow$ R, g: X $\rightarrow R^{n_g}$ and h: X $\rightarrow R^{n_h}$ are defined on X.

The function f is usually called the objective function. Each of the constraints $g_i(x) \geq 0, i = 1, \ldots, n_g$ is called an inequality constraint, and each of the constraints $h_i(x) \geq 0, i = 1, \ldots, n_h$, is called an equality constraint. Note also that the set X typically includes lower and upper bounds on the variables. The reason for separating variable bounds from the other inequality constraints is that they can play a useful role in some algorithms, i.e., they are handled in a specific way. A vector $x \in X$ satisfying all the constraints is called a feasible solution to the problem; the collection of all such points forms the feasible region. The NLP problem, then, is to find a feasible point x^* such that $f(x) \geq f(x^*)$ for each feasible point x. Needless to say, a nonlinear programming problem can be stated as a maximization problem, and the inequality constraints can be written in the form $g_i(x) \leq 0, i = 1, \ldots, n_g$.

In practice, linear programming assumptions or approximations may lead to appropriate problem representations over the range of decision variables being considered. The development of highly efficient and robust algorithms and software for linear programming, the advent of high-speed computers, have made linear programming an important tool for solving problems in diverse fields. However, many business processes behave in a nonlinear manner. Many realistic problems cannot be adequately represented or approximated as a linear programming due to the nature of the nonlinear of the objective function and/or the nonlinearity of any of the constraints. Efforts to solve such nonlinear problems efficiently have made rapid progress during the past four decades (Bazarra, 2006).Nonlinearities in the form of

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either nonlinear objective functions or nonlinear constraints are crucial for representing an application properly as a mathematical program. For example, the price of a bond is a nonlinear function of interest rate, and the price of a stock option is a nonlinear function of the price of the underlying stock. The marginal cost of production often decreases with the quantity produced, and the quantity demand for a product is usually a nonlinear function of the price. Other examples are inventory control, scheduling, blending, water pollution control, etc. Two detailed examples of portfolio selection and water resources planning are described below (Bertsekas, 1999).

Portfolio selection problem: An investor has \$5000 and two potential investments. Let x_j for $j = 1$ and $j=2$ denote his allocation to investment *j* in thousands of dollars. From historical data, investments 1 and 2 have an expected annual return of 20 and 16 percent, respectively. Also, the total risk involved with investments 1 and 2, as measured by the variance of total return, is given by $2x_1^2 + x_2^2 + (x_1 + x_2)^2$, so that risk increases with total investment and with the amount of each individual investment. The investor would like to maximize his expected return and at the same time minimize his risk. There are several possible approaches. For example, he can minimize risk subject to a constraint imposing a lower bound on expected return. Alternatively, expected return and risk can be combined in an objective function, to give the model:

Maximize

$$f(x) = 20x_1 + 16x_2 - \theta\left[2x_1^2 + x_2^2 + (x_1 + x_2)^2\right]$$

Subject to:

$$g_1(x) = x_1 + x_2 \leq 5,$$

$$x_1 \geq 0,\ x_2 \geq 0$$

The nonnegative constant θ reflects his tradeoff between risk and return. If θ =0, the model is a linear program, and he will invest completely in the investment with greatest expected return. For very large θ, the objective contribution due to expected return becomes negligible and he is essentially minimizing his risk. This is a simplified portfolio selection problem as a nonlinear programming example. It is not easy to define a reliable yield/risk formula. More research is needed to modify the model in practice.

Water resources planning problem: In regional water planning, sources emitting pollutants might be required to remove waste from the water system. Let x_j be the pounds of Biological Oxygen Demand (an often-used measure of pollution) to be removed at source *j.* One model might be to minimize total costs to the region to meet specified pollution standards:

$$\text{Minimize } \sum_{j=1}^{n} f_j(x_j),$$

Subject to:

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \qquad (i = 1, 2, \ldots\ldots m)$$

$$0 \leq x_j \leq u_j \qquad (j = 1, 2, \ldots\ldots n)$$

where

$f_j(x_j) =$ Cost of removing x_j pounds of Biological Oxygen Demand at source *j,*

$b_i =$ Minimum desired improvement in water quality at point *i* in the system,

$a_{ij} =$ Quality response, at point *i* in the water system, caused by removing one pound of Biological Oxygen Demand at source *j,*

$u_j =$ Maximum pounds of Biological Oxygen Demand that can be removed at source *j*