Utility Function

Gino Favero
University of Parma, Italy

INTRODUCTION

Since the dawning of Probability, the *expected value* had been unanimously regarded as a fair evaluation of a game with random payoff. Such a belief was drastically shaken by the *St. Petersburg paradox* (see below), and quickly led to realise that a better evaluation had to take into account a suitable "transformation" of the various payoffs, so as to mirror the "satisfaction" of the winner of the prize rather than the objective value of the prize itself. Such a transformation was called a *utility function*, and was soon used to build equilibrium and consumption models.

The early interpretations thought of utility as a true unit of measure of individual satisfaction, that allowed, for instance, to be summed within and across individuals; in this approach, it was necessary to postulate that an individual attached *a priori* a utility value to goods. Soon another line of thinking arose, suggesting that utility only captures rankings of the satisfaction levels of an individual choosing among several types of goods. This interpretation makes harder, for instance, to measure the overall utility of a population, but obtains a utility that can be deduced starting from *rationality axioms*, that is, from reasonable behavioural rules.

After providing some insights on the two approaches, and comparing them, a little overview of the applications of utility functions to decision making is given, along with some future trends.

BACKGROUND

The so-called "St. Petersburg paradox" was first raised by N. Bernoulli (1713), and can be shortly exposed as follows. Suppose that a game is proposed, based on repeatedly throwing a coin and paying, on the first occurrence of "heads," a prize that doubles at each previous occurrence of "tails." If, for instance, the base prize is "one bean," then one bean is paid if the first toss yields "tails," otherwise the prize becomes two beans, and so forth. In such a way, if the coin has to be tossed, say, five times before it yields the first occurrence of "tails," the prize paid at the fifth toss is $2^{5-1} = 16$ beans. It is straightforward that the expected value of the win is infinite; yet, and hence the term "paradox," it seems unrealistic that a rational investor would accept to trade his entire wealth with the right of playing such a game a single time.

A few years later, Cramer (1728) and D. Bernoulli (1738) proposed two solutions to such a problem. Although apparently rival to each other, both their approaches were based on the idea that a player would focus, rather than on (the expected value of) the "material satisfaction" measured by the monetary value of a win, on the "moral satisfaction" which is supposed to be a mathematical function of that win. Such function was the square root for Cramer and the logarithm for D. Bernoulli: notably, both of them were increasing and concave, or marginally decreasing (that is to say, with a decreasing derivative function).

DOI: 10.4018/978-1-4666-5202-6.ch232

The actual development of the concept of marginal utility had to wait until the second half of the 19th century, when it was almost simultaneously proposed by Jevons (1871), Menger (1871). and Walras (1874). Their works, implying the need to abandon the concept of "objective" value of goods in favor to a "subjective" value essentially determined by the equilibrium of supply and demand, can be considered to mark the transition from Classical to Neoclassical economics.

In the spirit of Bentham (1776) and Mill (1863), the utility they had in mind was a *cardinal*, "objective" unit of measure for people's satisfaction. For instance, if an individual's utility for a cup of tea and a glass of milk were, respectively, 10 and 5, it would be acceptable to conclude that such an individual values tea "twice as" milk, and sometimes even that (s)he would consider "equivalent" a cup of tea and *two* glasses of milk. Moreover, utilities could even be added across individuals in order to obtain the "total amount" of satisfaction of a community, whose maximisation had to be the main concern of political choices.

Some criticism to such a model came from the observation that it seemed unrealistic for a person to attach a numerical evaluation to every object or amount of money and then act "mechanically" as maximisers of such an evaluation. Indeed, at this stage, utility was a *normative* tool, which had to be assumed by individuals in order to allow for economical modelisation.

MAIN FOCUS

Pareto's *Ophelimity* and Cantor's Theorem

Another strong criticism to cardinal utility was raised by Pareto (1906), who claimed that only a "subjective" numerical evaluation of the desirability of goods for a single individual was possible, and only up to the purpose of ranking her or his own preferences: the utility cannot be cardinal, but just *ordinal*. To underline the concept, he also proposed to replace the term "utility" (which he deemed to be too morally connoted) with the term *ophelimity* (coming from the greek "desirable").

Pareto's ordinal utility approach also allows to overcome the objection about "mechanicity" of utility as such. Suppose indeed that an individual is called upon expressing preferences over a (finite or countable, although possibly multidimensional) set of alternatives, which we shall call X, and write $x \succcurlyeq y$ to indicate that the individual prefers (possibly weakly, i.e., admitting indifference) the alternative $x \in X$ over $y \in X$. A result of Cantor (1915) can be easily adapted to show that such a set of preferences is *complete* (*i.e.*, the decision maker is never undecided, or, in mathematical terms, however given $x, y \in X$, it is either $x \succcurlyeq y$ or $y \succcurlyeq x$) and *transitive* (*i.e.*, $x \succcurlyeq y$ and $y \succcurlyeq z$ imply $x \succcurlyeq z$) if and only it can be *represented* by an "ophelimity" (utility) function, meaning that there exists a function $u\colon X \to \mathbb{R}$, associating an "opelimity value" to every alternative $x \in X$, such that $x \succcurlyeq y$ if and only if $u(x) \geq u(y)$.

This way, ophelimity functions are no longer *normative* tools, but become *descriptive*, that is, suitable to naturally describe the behaviour of an individual acting according to reasonable requirements (completeness and transitivity, above) or, shortly, a *rational individual*. On the other hand, every quantitative interpretation of such "ophelimity" is no longer possible, because every strictly increasing transformation of the utility function u still represents the same preferences. Such a description of individual preferences led to replacing the Benthamian utility maximisation criterion with the concept of Pareto optimality, according to which a system is enjoing optimal economic satisfaction if no one can increase her or his own utility without decreasing someone else's.

Utility According to von Neumann and Morgenstern

In their approach, marking the birth of Decision Theory, von Neumann and Morgenstern (1944) focused on the case when the set X of alternatives