Constrained Nonlinear Optimization in Information Science

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INTRODUCTION

A company manufactures new smart-phones that are supposed to capture the market by storm. The two main inputs components of the new smart-phone are the circuit board and the relay switches that make the phone faster and smarter and give it more memory.

The number of smart-phones to be produced is estimated to equal $E = 200x_1^{\frac{1}{3}}x_2^{\frac{1}{2}}$, where *E* is the number of smart-phones produced and x_1 & x_2 are the number of circuit broad hours and the number of relay hours worked, respectively. Such a function is known to economists as a *Cobb–Douglas function*. Laborers are paid by the type of work they do: the circuit boards and the relays for $5 and $10 an hour, respectively. We want to maximize the number of smart-phones to be made if we have $150,000 to spend on these components in the short run.

Lagrange multipliers can be used to solve *nonlinear optimization problems* (NLPs) in which all the constraints are equality constrained. We consider the following type of NLPs as shown by Equation (1):

Maximize (Minimize) $z = f(x_1, x_2, ..., x_n)$

$$\text{(1)}$$

Subject to

$$g_1(x_1, x_2, ..., x_n) = b_1$$
$$g_2(x_1, x_2, ..., x_n) = b_2$$
$$...$$
$$g_m(x_1, x_2, ..., x_n) = b_m$$

In our smart-phones example, we find we can build an equality constrained model. We want to maximize

$$E = 200x_1^{\frac{1}{3}}x_2^{\frac{1}{2}}$$

subject to the equality constraint

$$5x_1 + 10x_2 = 150,000$$

Problems in information science and technology such as this can be modeled using constrained optimization. We begin our discussion with equality constrained optimization, then discuss the inequality constrained optimization, and finally discuss some numerical methods to approximate the solutions.

BACKGROUND

The general constrained nonlinear programming (NLP) problem is to find x* as to optimize $f(X), X = (x_1, x_2, ..., x_n)$ subject to the constraints of the problem shown in equation (2).

Maximize or Minimize $f(x_1, x_2, ..., x_n)$

subject to

$$g_i(x_1, x_2, ..., x_n) \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} b_i \qquad \text{(2)}$$

for i = 1,2,…,m.

Classical constrained optimization appeared with equality constrains and Lagrange multiplier named for Joseph Lagrange in the late 1700's. It was almost two hundred years later when Kuhn-Tucker (1951) presented their famous Kuhn-Tucker (KT) conditions. Scholars later found that Karsuh (1939) had done considerable work on his thesis in the area of constrained optimization and thus, was added his name was added to create the Karsh-Kuhn-Tucker (KKT) conditions. Bellman (1952; 1957) created dynamic programming in the 1950's to handle sequential constraints in optimization.

Methods to ease computation and complexity of certain forms of NLP were created to ease the solving of such problems. Quadratic programming (Bazaraa et al., 1993) and Wolfe's method (1959) showed how to convert some NLPs into linear programs. Separable programing was another heuristic procedure described by Winston (2002) and Bazaraa et al. (1993). Complexities in problems have led to other heuristic models. The evolutionary strategy optimization technique was created in the early 1960s and developed further in the 1970s and later by Ingo Rechenberg, Hans-Paul Schwefel and his co-workers. Further development was done in evolutionary strategies by Michalewizc (1999). Strides in computer science led to the development of genetic algorithms, (Homaifar et al. 1994; Stender, 1994; Burke et al., 2005) and then to particle swarm optimization (Kennedy, 1995; 1997; Parsopolous et al., 2002). The complexity of the problems dictates the methods to be used. Cuckoo search is the most recent optimization method (Yeng et al, 2009)

MAIN FOCUS

Introduction and Basic Theory

To solve NLPs in the form of Equation (2), we associate a Lagrangian multiplier, λ_i, with the ith constraint and form the Lagrangian equation to get Equation (3):

$$L(X, \lambda) = f(X) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(X)) \qquad (3)$$

The computational procedure for Lagrange multipliers requires that all the partials of this Lagrangian function, Equation (4), must equal zero. These partials are the *necessary conditions* of the NLP problem. These are the conditions required for $\mathbf{x} = \{x_1, x_2,..., x_n\}$ to be a solution to Equation (3).

The Necessary Conditions   (4)

$\partial L / \partial Xj = 0$ ($j = 1, 2,..., n$ variables)   (4a)

$\partial L / \partial \lambda_i = 0$ ($i = 1, 2,..., m$ constraints)   (4b)

Definition: $\mathbf{x}$ is a regular point if and only if $\nabla g_i(x)$, $i = 1, 2,..., m$ are linearly independent.

Theorem 1

   a.   Let (2) be a maximization problem. If f is a concave function and each $g_i(x)$ is a linear function, then any point satisfying Equation (4) will yield an optimal solution.
   b.   Let (2) be a minimization problem. If f is a convex function and each $g_i(x)$ is a linear function, then any point satisfying Equation (4) will yield an optimal solution.

The Hessian matrix is used to determine if a function was convex, concave, or neither. We also note that the above theorem limits our constraints to linear functions. What if we have nonlinear equality constraints?

We can use the bordered Hessian in the sufficient conditions. Given the bivariate Lagrangian function as in

$$L(x_1, x_2, \lambda_i) = f(x_1, x_2) + \sum_{i=1}^{m} \lambda_i (b_i - g(x_1, x_2))$$

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