An Essay on Denotational Mathematics

Giuseppe Iurato
University of Palermo, Italy

INTRODUCTION

Denotational Mathematics (in short, DM) is a new discipline which serves as a formal language to lay out and solve the various problems posed, amongst others, by cognitive informatics, software science and computational intelligence. It has been worked out in the last decades by Yingxu Wang (2008a-e, 2009a-c, 2010a-b, 2011, 2012a-b)[1], to whose many works and papers we refer for a wider and more complete information science contextualization of DM and its applications.

In this contribution, we wish to briefly outline, but in a rigorous manner, the main formal structures considered by DM, within the framework of classical algebra of relations. Our exposition is extremely technical and synthetic to allow a faster and clearer overall view of the pivotal mathematical structures of DM, which are *concept algebra*, *real-time process algebra*, *system algebra*, *visual semantic algebra*, *granular algebra*, and *inference algebra*.

In relation to its possible applications to the general context of information sciences, numerical mathematics, as regard continuous context, mainly deals with rational ($\mathbb{Q}$), real ($\mathbb{R}$) and complex ($\mathbb{C}$) number systems, while, as regard discrete context, it deals with natural ($\mathbb{N}$), integral ($\mathbb{Z}$) and binary ($\mathbb{B}$) numerical systems. DM may be considered either from the classical algebra viewpoint (as done here) or from the model theory one[2]. In any case, it mainly deals with *hypercomplex* ($\mathbb{H}$) denotational algebraic systems, which will be defined in the next section 4.

BACKGROUND

Generally speaking, methods of mathematics may be roughly classified into *analytical*, *numerical*, and *denotational* approaches. The analytical approach deals with continuous mathematical objects defined in the real number field ($\mathbb{R}$); the numerical approach deals with discrete mathematical entities defined in $\mathbb{R}$ and in binary systems ($\mathbb{B}$) with iteratively and recursively approximation functions.

However, the domains of recent problems in the emerging disciplines of artificial intelligence, cognitive informatics, brain science, cognitive computing, knowledge science, and computational intelligence require rather new *complex hyperstructures* ($\mathbb{H}$), beyond that of pure numbers in ($\mathbb{B}$) and ($\mathbb{R}$), which just identify the denotational approach.

The requirement for reducing complicated formal problems and solutions to the low-level objects in conventional computing theories and the underpinning of analytical or numerical mathematical means, has greatly constrained the inference and computing ability of both human and intelligent systems.

This has motivated the introduction of transdisciplinary investigations provided by *denotational approach* based on novel mathematical structures for modelling and processing new complex hyperstructures required to be established by cognitive computing and computational intelligence, collectively known just as denotational mathematics (DM).

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DM may be viewed as a rigorous formal approach to processing both complex architectures and intelligent formal behaviours in order to cope with the fundamental challenges in understanding, describing, and modelling natural and machine intelligence in general, as well as concept, knowledge and semantics, in particular.

As a counterpart of the classic analytic mathematics, DM encompasses new forms of mathematical structures for dealing with complex objects recently emerged in a very wide range of contemporary disciplines and fields such as cognitive informatics, cognitive computing, abstract intelligence, artificial intelligence, computational intelligence, semantic computing, software science, knowledge science, system science and computing with words.

In recognizing mathematics as the main meta-methodology of all informatics sciences and engineering disciplines[3], a set of DM structures have been created in the last decades. DM provides a coherent set of contemporary mathematical means and explicit expressive power for dealing with both complex mathematical objects and long-chains of serial and embedded mathematical operations emerged in multiple modern disciplines.

In DM, not only the mathematical entities are greatly complicated at a formal level, but also the mathematical structures and methodologies are significantly expanded from simple relations and independent functions to embedded relations and serially coupled functions.

In what follows, we outline the chief elements and aspects of the main DM structures, with a view toward their possible applications to information sciences and technology.

THE NOTION OF HYPERCOMPLEX ALGEBRAIC SYSTEM

Generally speaking, an *algebraic system* is a tuple $\Xi \overset{def}{=} \left(\{A_i\}_{i\in I};\{R_j\}_{j\in J}\right)$, where A_i is a non-empty set for each $i\in I$, and $\phi \neq R_j \subseteq \Pi_{k\in K_j} A_k$ for each $j\in J$, is a *relation*. If, for instance, $I=\{1,2\}$ and J is a singleton, then $\phi \neq R \subseteq A_1 \times A_2$ is a *binary* (*external*) *relation* with *domain* in A_1 and *codomain* in A_2; if $A_1=A_2=A$, then R is a *binary* (*internal*) *relation* on the *support* A.

For example, a *semigroup* is an algebraic system of the type $(A;R)$, where A is a non-empty set (*support*) and R is a ternary internal relation $\phi \neq R \subseteq A^3$ such that

$$\begin{array}{l}(a,b,c) \in R \wedge (a,b,c') \in R \\ \quad \Rightarrow (c = c')\forall a,b,c,c' \in A\end{array} \quad (\textit{univocity of } R),$$

which verifies the associative law. Often, such a ternary internal relation is written as a function of the type $R: A\times A\rightarrow A$, with $(a,b)\in R$ wrote as aRb.

An algebraic subsystem (or restriction) ž' of Ξ, is an algebraic system of the type ž' $\overset{def}{=} \left(\left\{A_i'\right\}_{i\in I};\left\{R_j'\right\}_{j\in J}\right)$, where $\varnothing \neq A_i' \subseteq A_i \quad \forall i \in I$, and $R_j' \subseteq \prod_{k\in K_j'} A_k'$ with $R_j' = R_j \mid_{\prod_{k\in K_j'} A_k'}$ and $K_j' \subseteq K_j$ for each $j\in J$. Likewise, starting from ž', Ξ may be considered as an algebraic extension of ž'. The notion of action of a set G on another set A, may be generalized considering it as an external relation of the type $R \subseteq G \times A$ (Artin, 1991).

Let $\mathcal{F} = \left(\left\{A_i\right\}_{i\in I};\left\{R_j\right\}_{j\in J}\right)$ be an algebraic system. A denotational algebraic hypercomplex system $\mathbb{H}_{\mathcal{F}}$ in the framework $\mathcal{F}$, is an algebraic systems of the type $\mathbb{H}_{\mathcal{F}} \overset{def}{=} \left(\left\{B_k\right\}_{k\in K};\left\{S_l\right\}_{l\in L}\right)$, where $K=K_1\cup K_2$, $L=L_1\cup L_2$, with $K_1 \subseteq I, L_1 \subseteq J$, $K_1 \cap K_2 = L_1 \cap L_2 = \phi$, $B_k\in\{A_i\}_{i\in I}$ for each $k\in K$, $S_l\in\{R_j\}_{j\in J}$ for each $l\in L_1$, $\left(\bigcup_{k\in K_1} B_k\right) \cap \left(\bigcup_{k\in K_2} B_k\right) = \phi$, and $S_l \subseteq \prod_{k\in K_1^l \cup K_2^l} B_k$ with $K_1^l \subseteq K_1, K_2^l \subseteq K_2$ for each $l\in L_2$. Each S_l, $l\in L_1$, is said to be an internal relation of $\mathbb{H}_{\mathcal{F}}$, while

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