Chapter 1
Epistemological Notes on Mathematics

ABSTRACT

This chapter is an attempt to show how mathematical thought has changed in the last two centuries. In fact, with the discovery of the so-called non-Euclidean Geometries, mathematical thinking changed profoundly. With the negation of the postulate for "antonomasia," that is the uniqueness of the parallel for Euclid, and the construction of a geometric theory equally valid on the logical and coherence plane, called non-Euclidean geometry, the meaning of the word "postulate" or "axiom" changes radically. The axioms of a theory do not necessarily have to be dictated by real evidence. On this basis the constructions of arithmetic and geometry are built. The axiomatic-deductive method becomes the mathematical method. It will also highlight the constant link between mathematics and the reality that surrounds us, which tends to make itself explicit through an artificial, abstract language and with clear and certain grammatical rules. Finally, you will notice the connection with the existing technology, that is the new electronic and digital technology.

"Mathematics is the main language of science and technology, as such, it is the key to understanding and shaping the world around us." ~C. F. Gauss

1. EVOLUTION OF MATHEMATICAL THOUGHT IN THE NINETEENTH CENTURY

Mathematics is often considered a rigid Science, closed in its methods and in its structure. On the contrary, it is a living science, which changes with the very rapid evolution of scientific thought and technological progress. Indeed Mathematics is the crux of all science, in the modern conception of the term. In support of this statement, we can cite some of the greatest thinkers of the past.

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For example, in his great work "*Il Saggiatore*" (Galilei, 1864), Galileo Galilei writes: ""Philosophy [nature] is written in that great book which ever is before our eyes -- I mean the universe -- but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth."

In the last two centuries, mathematical thought has changed deeply. Until the end of the eighteenth century, Mathematics was considered as a science defined by its objects, by its contents. Today we can also find in part this attitude in those who think they can define Mathematics in the same way as philosophers and scientists of the 18th century did. At that time, Mathematics was considered as the "science of numbers" or the "science of quantities" (Boyer, 1968). Where the word "quantity" was meant both as "continuous quantity" and "discrete quantity". In the first case, it should have been part of geometry, in the second a part of arithmetic.

One of the most important results of the evolution which has occurred in the past two centuries is precisely that of having overcome the attitude of defining mathematics through its objects and so focusing the attention of researchers on the procedures and structures of the discipline. In this way, Mathematics has greatly expanded its domain of action, incorporating problems that concern, for example, formal logic, the theory of choices and decisions, computation and so on.

2. THE CRISIS OF GEOMETRY AND THE BIRTH OF GEOMETRIES

During the nineteenth century one of the main moments of the evolution of Mathematics was certainly the invention of the non-Euclidean Geometries by Gauss (1777-1855), Schwelkart (1778-1859), Lobachevski (1792-1856), Bolyal (1802-1860) and Riemann (1826-1866), together with the demonstration of their intrinsic coherence (Boyer, 1968). The first logical consequence was the radical change in the way in which we conceive of Geometry. In fact, if we suppose that there exists an entity external to us, which is the object of this science, it could not be described or studied with contradictory doctrines among them, such as Euclidean Geometry and any of the non-Euclidean ones. This is of course rests on the principle that every scientific theory is based on the internal coherence of the reality being studied. To better understand what we want to affirm, it is opportune to reflect on the *"Elements"* of Euclid (Fitzpatrick, 2007), the first scientific treatise in the history of humanity. It is a sequence of propositions; some of which describe the entities that will be discussed later. Other propositions express some truths that are considered incontrovertible and for this reason Euclid calls them "common notions". Finally, other propositions are presented as requests for consent, which is why they are called "postulates". The postulates concern properties of geometrical entities, or of the operations that we can do on them. These include the so-called postulate "par excellence" which basically affirms the uniqueness of a parallel to a given straight line for a point outside of it. Statements, common notions and postulates are given without proof while the subsequent propositions are rigorously demonstrated with procedures and reasoning based on the previously expressed statements.

For centuries, the *"Elements"* was considered paradigmatic for any scientific treatise which involved Mathematics. For example, the structure of Isaac Newton's famous work entitled "Philosophiae naturalis principia mathematica" was inspired by the Elements (Newton, 1883). It must be added that the "common notions" and the "postulates" have been considered, over the centuries, acceptable propositions because