Formal Concept Analysis Based Clustering

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INTRODUCTION

Formal concept analysis (FCA) is a branch of applied mathematics with roots in lattice theory (Wille, 1982; Ganter & Wille, 1999). It deals with the notion of a concept in a given universe, which it calls context. For example, consider the context of transactions at a grocery store where each transaction consists of the items bought together. A concept here is a pair of two sets (A, B). A is the set of transactions that contain all the items in B and B is the set of items common to all the transactions in A. A successful area of application for FCA has been data mining. In particular, techniques from FCA have been successfully used in the association mining problem and in clustering (Kryszkiewicz, 1998; Saquer, 2003; Zaki & Hsiao, 2002). In this article, we review the basic notions of FCA and show how they can be used in clustering.

BACKGROUND

A fundamental notion in FCA is that of a context, which is defined as a triple (G, M, I), where G is a set of objects, M is a set of features (or attributes), and I is a binary relation between G and M. For object g and feature m, gIm if and only if g possesses the feature m. An example of a context is given in *Table 1*, where an "X" is placed in the i^{th} row and j^{th} column to indicate that the object in row i possesses the feature in column j.

The set of features common to a set of objects A is denoted by $\beta(A)$ and is defined as $\{m \in M \mid gIm\ \forall g \in A\}$. Similarly, the set of objects possessing all the features in a set of features B is denoted by $\alpha(B)$ and is given by $\{g \in G \mid gIm\ \forall m \in B\}$. The operators α and β satisfy the assertions given in the following lemma.

Lemma 1 (Wille, 1982): Let (G, M, I) be a context. Then the following assertions hold:

1. $A_1 \subseteq A_2$ implies $\beta(A_2) \subseteq \beta(A_1)$ for every $A_1, A_2 \subseteq G$, and $B_1 \subseteq B_2$ implies $\alpha(B_2) \subseteq \alpha(B_1)$ for every $B_1, B_2 \subseteq M$.
2. $A \subseteq \alpha(\beta(A)$ and $A = \beta(\alpha(\beta(A))$ for all $A \subseteq G$, and $B \subseteq \beta(\alpha((B))$ and $B = \alpha(\beta(\alpha\ (B)))$ for all $B \subseteq M$.

Table 1. A context excerpted from (Ganter, and Wille, 1999, p. 18). a = needs water to live; b = lives in water; c = lives on land; d = needs chlorophyll; e = two seeds leaf; f = one seed leaf; g = can move around; h = has limbs; i = suckles its offsprings.

		a	b	c	d	e	f	g	h	i
1	Leech	X	X					X		
2	Bream	X	X					X	X	
3	Frog	X	X	X				X	X	
4	Dog	X		X				X	X	X
5	Spike-weed	X	X		X		X			
6	Reed	X	X	X	X		X			
7	Bean	X		X	X	X				
8	Maize	X		X	X		X			

A formal concept in the context (G, M, I) is defined as a pair (A, B) where $A \subseteq G$, $B \subseteq M$, $\beta(A) = B$, and $\alpha(B) = A$. A is called the extent of the formal concept and B is called its intent. For example, the pair (A, B) where A = {2, 3, 4} and B = {a, g, h} is a formal concept in the context given in *Table 1*. A subconcept/superconcept order relation on concepts is as follows: $(A_1, B_1) \leq (A_2, B_2)$ iff $A_1 \subseteq A_2$ (or equivalently, iff $B_2 \subseteq B_1$). The fundamental theorem of FCA states that the set of all concepts on a given context is a complete lattice, called the concept lattice (Ganter & Wille, 1999). Concept lattices are drawn using Hasse diagrams, where concepts are represented as nodes. An edge is drawn between concepts C_1 and C_2 iff $C_1 \leq C_2$ and there is no concept C_3 such that $C_1 \leq C_3 \leq C_2$. The concept lattice for the context in Table 1 is given in Figure 1.

A less condensed representation of a concept lattice is possible using reduced labeling (Ganter & Wille, 1999). Figure 2 shows the concept lattice in Figure 1 with reduced labeling. It is easier to see the relationships and similarities among objects when reduced labeling is used. The extent of a concept C in *Figure 2* consists of the objects at C and the objects at the concepts that can be reached from C going downward following descending paths towards the bottom concept. Similarly, the intent of C consists of the features at C and the features at the concepts that can be reached from C going upwards following ascending paths to the top concept.

Consider the context presented in *Table 1*. Let B = {a, f}. Then, $\alpha(B)$ = {5, 6, 8}, and $\beta(\alpha(B))$ = b({5, 6, 8}) = {a, d, f} $\neq$ {a, f}; therefore, in general, $\beta(\alpha(B)) \neq B$. A set of features B that satisfies the condition $b(\alpha(B)) = B$ is called a closed feature set. Intuitively, a closed feature set is a maximal set of features shared by a set of objects. It is easy to show that intents of the concepts of a concept lattice are all closed feature sets.

The support of a set of features B is defined as the percentage of objects that possess every feature in B. That is, support(B) = $|\alpha(B)|/|G|$, where $|B|$ is the

Figure 1. Concept lattice for the context in Table 1

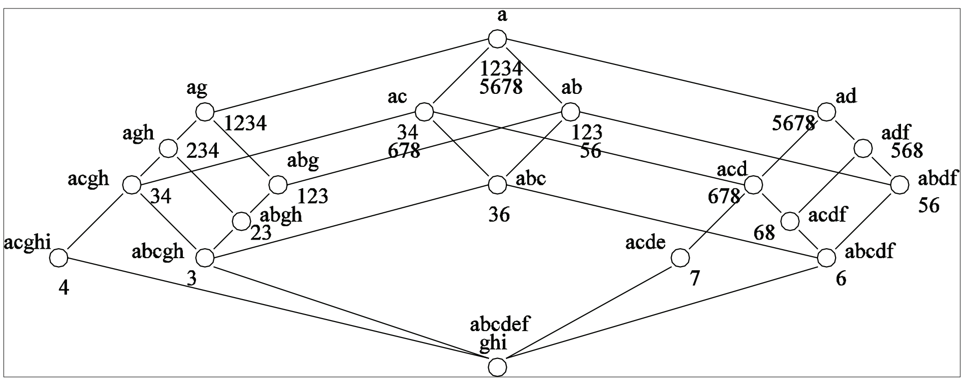
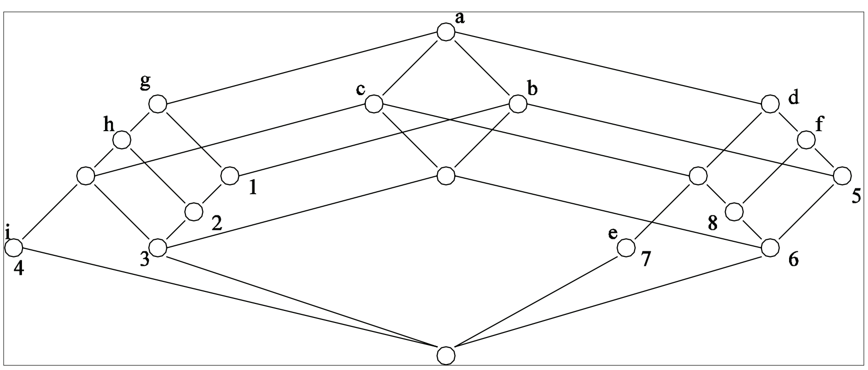


Figure 2. Concept lattice for the context in Table 1 with reduced labeling



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