Chapter 4
Visualization and Mathematical Thinking

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ABSTRACT

Drawing is not proving. For a long time, this argument has been used to avoid the use of visualization in mathematics. Nevertheless, a number of proofs, concepts, and ideas are easier to understand with the help of a small drawing. In this chapter, the author shows that visualization in mathematics is helpful not only to illustrate but also to create ideas, and this at all levels.

INTRODUCTION

The scene took place at the time of my studies. My professor was at the blackboard, in a packed lecture hall. In front of his fascinated students, he was proving a deep theorem of geometry, using number of diagrams he drew with confidence. The blackboard was becoming white but, suddenly, he stopped in the middle of a diagram (see Figure 1).

As time went by, the professor looked more and more puzzled. After a while, he started to make a little drawing, unfortunately hidden by his body. Suddenly, he looked illuminated, erased his drawing and resumed his proof with number of diagrams; we noted them without understanding well. At the end of the lecture, the bravest students went to the desk to ask him some explanations about his little drawing. His reply was unequivo-

Figure 1. Hervé Lehning, a diagram (© 2014, H. Lehning, New Math Diagram. Used with permission)

$$\begin{tikzcd}
\mathrm{H}\,(X, A) \arrow[r] \arrow[d, "q"] & \mathrm{H}\,(X, V) \arrow[d, "q"] & \mathrm{H}\,(X{-}A, V{-}A) \arrow[l] \arrow[d, "q"] \\
\mathrm{H}\,(X/A, V/A) \arrow[r] & \mathrm{H}\,(X, & {}
\end{tikzcd}$$

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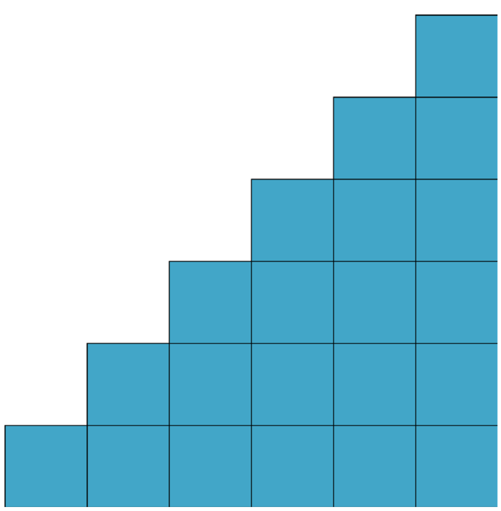
cal: "there's no question to fill your spirit with bad habits of thought". His reason to refuse was his pedagogical ideas: we must be freed of the errors of the past, and among them of the habit of using drawings to help intuition.

PROOFS WITHOUT WORDS

This conception of mathematics was dominant at the age of what was called "modern mathematics" or "new math" (Adler, 1972). Nevertheless, number of results has visual proofs (Nelsen, 1997). The simplest of them is probably the calculation of the sum of the first natural numbers as: 1 + 2 + 3 + 4 + 5. Of course, in this case, we find 15 easily but it will be more difficult to compute: 1 + 2 + 3 + 4 + … + 100. The general case: 1 + 2 + 3 + 4 + … + n is even more complex. The idea to compute it easily is to model this sum as the area of a staircase (see Figure 2).

By copying the staircase upside-down, we get a rectangle (see Figure 3).

Figure 2. Hervé Lehning, 1 + 2 + 3 + 4 + 5 + 6 (© 2014, H. Lehning. Used with permission)



Thus, twice the sum: 1 + 2 + 3 + 4 + 5 + 6 equals the area of the rectangle with side-lengths 6 and 7, which is 42, thus: 1 + 2 + 3 + 4 + 5 + 6 = 21. For the same reason:

$$1+2+3+\cdots+n=\frac{n\left(n+1\right)}{2}.$$

Today, this kind of proofs without words is generally accepted when they concern natural numbers (that is to say: 1, 2, 3, *etc*.). The same technique allows us to prove that the sum of the first n odd numbers equals the square of n (see Figure 4).

The identity: $(a + b)^2 = a^2 + 2\,ab + b^2$ has a proof that, *a priori*, looks of the same kind. If the sides of the blue and orange squares are a and b, their areas equal a^2 and b^2 while those of the green rectangles equal the product ab (see Figure 5).

However, this proof without words is different because, rigorously, it is correct only if a and b can be consider as lengths, that is to say are positive numbers. It is the main objection of the purists. However, we can make this proof rigorous, but at the price of the use of sophisticated mathematics. For that, we consider the difference: $f(x, y) = (x$

Figure 3. (Hervé Lehning, 1 + 2 + 3 + 4 + 5 + 6 twice (© 2014, H. Lehning. Used with permission)