Chapter 8
A Different Way to Look at Random Variables:
Do Decision Makers Look at Functions or at Sets?

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ABSTRACT

A classical problem in Decision Theory is to represent a preference preorder among random variables. The fundamental Debreu's Theorem states that, in the discrete case, a preference satisfies the so-called Sure Thing Principle if and only if it can be represented by means of a function that can be additively decomposed along the states of the world where the random variables are defined. Such a representation suggests that every discrete random variable may be seen as a "histogram" (union of rectangles), i.e., a set. This approach leads to several fruitful consequences, both from a theoretical and an interpretative point of view. Moreover, an immediate link can be found with another alternative approach, according to which a decision maker sorts random variables depending on their probability of outperforming a given benchmark. This way, a unified approach for different points of view may be achieved.

INTRODUCTION

A highly innovative result about the representation of a given preference on a set of random variables is due to Debreu (1960). By replacing the classical Independence Axiom of von Neumann and Morgenstern (1944) with the Sure Thing Principle due to Savage (1951), and using topological techniques, he proved that, in the discrete case, a preference among random variables can be represented by means of a functional that can be additively decomposed with respect to the states of the world. Debreu's Theorem is very deep, both from a mathematical and an economical point of view; unfortunately, though,

DOI: 10.4018/978-1-4666-9458-3.ch008

it is still scarcely used in the economic literature and its interpretation has not yet been fully discussed (see, however, Mas-Colell, 1990; moreover, Nau, 2003 & 2006, proposed and application of Debreu's Theorem to Decision Theory). The reason is perhaps that Debreu's representation, featuring no utility and no probability at all, looks a little "heretical" and tough to fit in the mainstream of the research in Decision Theory.

The Measure Representation Approach, developed by Segal (1987a, 1987b, and 1993), Wakker (1993), and LiCalzi (1998) identifies random variables with sets, in a suitable way; this allows for several new insights in the interpretation of Debreu's Theorem. Moreover, a strong bridge is laid down with the approach of "benchmarking acts", proposed by Castagnoli (1990), Castagnoli and LiCalzi (1996) and Bordley and LiCalzi (2000), according to which the expected utility is replaced by the probability of overperforming a given random benchmark.

In this chapter, these lines of research are briefly exposed and developed, showing that the identification of random variables with sets gives a new flavour to Debreu's Theorem, and makes it possible to read the representing functional in full probabilistic terms: as a matter of fact, for the problem at stake, the language and the tools of Measure Theory are undoubtedly more powerful and efficient than those of Functional Analysis. Several extensions of Debreu's Theorem have indeed been proposed in the past years (see, in particular, Chew &Wakker, 1996, and Wakker&Zank, 1999), but, since they use the language of Functional Analysis, none of them could obtain the highest degree of generality; the goal was attained by Castagnoli and LiCalzi (2006) just because they used typical Measure Theory instruments instead of Functional Analysis ones (*i.e.*, working with sets, not with functions). Successively, Castagnoli and Favero (2010), inspired by Chateauneuf (1999), used the same methods to characterise a "reversed" version of the Sure Thing Principle, based on monetary amounts instead of states of the world.

The present state of the literature will be exposed, together with some comments and remarks that might develop some new insights. In the next section, the classical utility *à la* von Neumann and Morgenstern (1944) will be introduced, together with the Sure Thing Principle, Debreu's Theorem and some meaningful generalisations and interpretations. Successively, starting from Debreu's Theorem, we shall propose to look at random variables not as functions, defined on a set of states of the world, but rather as sets ("histograms"), and to measure them by means of a bidimensional measure. In the third part of the chapter, we shall present the utility models seen before in terms of histograms, and some perspective of the possible future extensions will be given.

BACKGROUND

A classical problem in Decision Theory is to *represent* a preference preorder among *lotteries*, *i.e.*, probability assessments over a given set of consequences, or *random variables*, *i.e.*, functions defined on some set Ω of "states of the world". In the latter case, it often occurs, though, that Ω is *not* the set the decision maker is primarily interested into: think, for instance, of the case when a decision maker wants to choose an investment in order to guarantee a serene future to her/himself and her/his family. Of course, the random events that matter to her/him are related with the needs of her/his family, with her/his salary, with the future health conditions of the family members, and so on. On the other hand, the financial investments that are available to the decision maker are linked to a completely different set of states of the world, which have to do with things such as government's policies, inflation level, status

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