Chapter 6
Optimization of Utility Functions in an Admissible Space of Higher Dimension

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ABSTRACT

S. Smale published a paper where announce a theorem which optimize a several utility functions at once (cf. Smale, 1975) using Morse Theory, this is a very abstract subject that require high skills in Differential Topology and Algebraic Topology. Our goal in this paper is announce the same theorems in terms of Calculus of Manifolds and Linear Algebra, those subjects are more reachable to engineers and economists whom are concern with maximizing functions in several variables. Moreover, the elements involved in our theorems are accessible to graduate students, also we putting forward the results we consider economically relevant.

1. INTRODUCTION

We introduce four definitions that are fundamental in the study of optimization of several utility functions into a pure exchange economy. The first of them is a *convex cone* defined in terms of the utility function; more precisely, we defined the convex cone like a subset in the tangent space of the commodity space. The definitions complete the assumptions of a pure exchange economy. *Curve that maximize the utility*, mathematically this is an argument used in differential topology and we introduced into the approach of the main problem, because is a strong argument in the proof of the main theorems. Moreover, the curve that maximize the utility function introduces an idea of a path that converge into the optimum point in

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the commodity space and we may thought this curve as a trade that lead the Pareto optimum point to our agents whose are playing in this economy. The *Pareto critical set*, is a more general definition that the one used in the classical theory, our definition have the needs to deals with the main problem, we also prove two statements one is a topological characterization of this set and the other is an analytical characterization of this set, both of them are very important in the proof of our theorems. The *Generalized Hessian* is the last definition we introduced, actually this is a framework in the theory of singularities is studied like a generalization of the classical definition and is applied into high dimensional manifolds, but we used to give an analogous condition of the second derivative used in elemental calculus. Finally we introduce two theorems, *Necessary conditions theorem* which is the analogous to find a critical point for a one variable function, up to in this case we are concerning with the Pareto critical set. The last statement is the *sufficient condition theorem* involves the generalized Hessian.

2. METHODS

2.1 Utility Function

We assume an economy with l numbers of commodities and the market of each commodity have defined a unity of weight for each good, i.e. each good is quantified and is related with a real number, we will denote the real number set with $\mathbb{R}$. We are considering only those commodities that have in the market, that mean we use only positive real numbers. Then we denote $\mathbb{R}^l_+ := \left\{ x \in \mathbb{R}^l \mid x > 0 \right\}$ the orthant positive of $\mathbb{R}^l$ (the real space of dimension l) and P to the commodity space, then there exist a homeomorphism of P with an open subset of $\mathbb{R}^l_+$ indeed there is a map

$i : P \to \mathbb{R}^l_+$, which is an *inclusion.*

The coordinate $x \in P$ is a commodity bundle that belongs to a consumer, or the consumer wish to choose between all the l commodities. Also assume m agents in the market, we denoted x_i to the commodity bundle of the *i*-agent, then exist $x_i \in P$ bundles for $i = 1, 2, ..., m$ agents. The status of the economy assuming the commodities of the m agents is denoted with the point $x \in P^m$; i.e. the complete economy of the m agents is represented with a point in the commodity space of dimension m . In our economic model we assume exhausted resources and those are quantified, then the next space is a natural definitions.

Definition 1: Let $p \in P$ the total commodity that agents are allow to choose, we named *attainable space* to the set

$$W = \left\{ x \in P^n \,\middle|\, \sum_{i=1}^{n} x_i = p \text{ with } x = \left(x_1, x_2, \ldots, x_n\right) \right\},$$

In the above space there are interacting the m agents and it represent the whole space of consumption or choice possibilities.

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