# Chapter 22
# The Hyper-Zagreb Index and Some Properties of Graphs

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## ABSTRACT

*Let G=(V(G),E(G)) be a graph. The forgotten topological index of G, denoted F(G), is defined as* $\sum_{v\in V(G)} (d(v))^3$. *The second Zagreb index of G, denoted* $M_2(G)$, *is defined as* $\sum_{uv\in E(G)} d(u)d(v)$. *The hyper-Zagreb index of G, denoted HZ(G), is defined as* $F(G)+2M_2(G)$. *The authors consider only finite undirected graphs without loops or multiple edges. Lots of theorems have been presented along with related results. Using the hyper-Zagreb index of the complement of a graph, they present sufficient conditions for some properties of the graph.*

## INTRODUCTION

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in (Bondy et al, 1976). Let $G = \left(V\left(G\right), E\left(G\right)\right)$ be a graph. We use $n$, $e$, $\delta$, and $\kappa$ to denote the order, size, minimum degree, and connectivity of $G$, respectively. The complement of $G$ is denoted by $G^c$. We also use $K_n$ and $E_n$ to denote the complete graph and the empty graph of order $n$. The forgotten topological index of $G$, denoted $F(G)$, is defined as $\sum_{v\in V(G)} (d\left(v\right)$ $\left(Furtlua\,et\,al, 2015\right)$. The second Zagreb index of $G$, denoted $M_2(G)$, is defined as $\sum_{uv\in E(G)} d\left(u\right)d\left(v\right)$ (see (Gutman et al, 1975)). The hyper-Zagreb index of $G$, denoted $HZ(G)$, is defined as $F(G)+2M_2(G)$ (see (Milovanovic et al, 2019). We use $\mu_n(G)$ to denote the largest eigenvalue of the adjacency matrix of a graph $G$ of order $n$. For two disjoint graphs $G_1$ and $G_2$, we use $G_1+G_2$ and $G_1 \vee G_2$ to denote respectively the union and join of $G_1$ and $G_2$. The concept of closure of a graph $G$ was introduced by Bondy and Chvátal in (Bondy et al, 1976). The $k$-closure of a graph $G$, denoted $cl_k(G)$, is a graph obtained from $G$ by recursively joining two nonadjacent vertices such that their degree sum is at least $k$ until no such pair remains. We use $C(n,r)$

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to denote the number of $r$-combinations of a set with $n$ distinct elements. A cycle $C$ in a graph $G$ is called a Hamiltonian cycle of $G$ if $C$ contains all the vertices of $G$. A graph $G$ is called Hamiltonian if $G$ has a Hamiltonian cycle. A path $P$ in a graph $G$ is called a Hamiltonian path of $G$ if $P$ contains all the vertices of $G$. A graph $G$ is called traceable if $G$ has a Hamiltonian path. A graph $G$ is $k$-Hamiltonian if for all $X \subset V\left(G\right)$ with $|X| \leq k$, the subgraph induced by $V(G)$-$X$ is Hamiltonian. Clearly, G is 0-Hamiltonian if and only if $G$ is Hamiltonian. A graph $G$ is $k$-edge-Hamiltonian if any collection of vertex-disjoint paths with at most $k$ edges is in a Hamiltonian cycle in $G$. Clearly, G is 0-edge-Hamiltonian if and only if $G$ is Hamiltonian. A graph $G$ is $k$-path-coverable if $V(G)$ can be covered by $k$ or fewer vertex-disjoint paths. Clearly, G is 1-path-coverable if and only $G$ is a traceable. A graph $G$ is $k$-connected if it has more than $k$ vertices and $G$ is still connected whenever fewer than $k$ vertices are removed from G. A graph $G$ is $k$-edge-connected if it has at least two vertices and $G$ is still connected whenever fewer than $k$ edges are removed from G.

The following results were obtained by Fiedler and Nikiforov.

**Theorem 1:** Let $G$ be a graph of order $n$.

If $\mu_n\left(G^c\right) \leq \sqrt{n-1}$, then $G$ contains a Hamiltonian path unless $G=K_{n+1}+v$.

If $\mu_n\left(G^c\right) \leq \sqrt{n-2}$, then $G$ contains a Hamiltonian cycle unless $G=K_{n+1}+e$.

Using the ideas and techniques developed by Fiedler and Nikiforov (2010), Li (2019) obtained sufficient conditions which involve the hyper-Zagreb indexes of the complements of the graphs for the Hamiltonian and traceable graphs. It is found that the ideas and techniques in (Li et al, 2019) can also be utilized to obtain sufficient conditions based upon hyper-Zagreb indexes for the additional properties of graphs. The aim of this paper is to present those conditions for $k$-Hamiltonian, $k$-edge-Hamiltonian, $k$-path-coverable, $k$-connected, and $k$-edge-connected graphs. The main results of this paper are as follows.

**Theorem 2:** Let $G$ be a graph of order $n \geq k+6$, where $k$ is an integer and $k \geq 1$. If

$$HZ\left(G^c\right) = F\left(G^c\right) + 2M_2\left(G^c\right) \leq (n-1)^2\left(n-k-2\right),$$

then $G$ is $k$-Hamiltonian or $G = K_{k+1} \vee \left(K_1 + K_{n-k-2}\right)$.

**Theorem 3:** Let $G$ be a graph of order $n \geq k+6$, where $k$ is an integer and $k \geq 1$. If

$$HZ\left(G^c\right) = F\left(G^c\right) + 2M_2\left(G^c\right) \leq (n-1)^2\left(n-k-2\right),$$

then $G$ is $k$-edge-Hamiltonian or $G = K_{k+1} \vee \left(K_1 + K_{n-k-2}\right)$.