Chapter 13
A Metaheuristic Approach to the Graceful Labeling Problem

Houra Mahmoudzadeh
Sharif University of Technology, Iran

Kourosh Eshghi
Sharif University of Technology, Iran

ABSTRACT

In graph theory, a graceful labeling of a graph G = (V, E) with n vertices and m edges is a labeling of its vertices with distinct integers between 0 and m inclusive, such that each edge is uniquely identified by the absolute difference between its endpoints. In this paper, the well-known graceful labeling problem of graphs is represented as an optimization problem, and an algorithm based on Ant Colony Optimization metaheuristic is proposed for finding its solutions. In this regard, the proposed algorithm is applied to different classes of graphs and the results are compared with the few existing methods inside of different literature.

GRACEFUL LABELING PROBLEM

In graph theory, a graceful labeling of a graph $G = (V, E)$ with n vertices and m edges is a labeling of its vertices with distinct integers between 0 and m inclusive, such that each edge is uniquely identified by the absolute difference between its endpoints. To be more precise, if we assume $G = (V, E)$ to be an undirected graph without loops or double connections between vertices, a graceful labeling of G, with n vertices and edges, is a one-to-one mapping f of the vertex set V into the set $\{0, 1, 2, ..., m\}$, so that if we assign an edge label $| f(x) - f(y) |$ to any edge (x, y), each edge receives a distinct positive integer label. A graph that can be gracefully labeled is called a graceful graph (Rosa, 1967). A sample graceful graph is shown in Figure 1. Vertex labels are shown inside the vertex circles, and edge labels are shown in red near the related edges.

The name "graceful labeling" is due to Solomon W. Golomb; however, this class of labelings was

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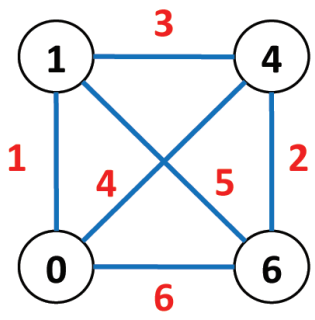
Figure 1. An example of graceful labeling of a graph



originally given the name β-labelings by Alex Rosa in a 1967 paper on graph labeling (Rosa, 1967). The computational complexity of the graceful labeling problem is not known, but a related problem called harmonious labeling was shown to be NP-complete (Gallian, 2009). In fact, the graceful labeling problem is rather a well-known example of the problems in NP, which are not known to be NP-complete, and neither known to be in P (Johnson, 2005). Many variations of graph labeling have been introduced in recent years by researchers. Various classes of graphs have been proven mathematically to be graceful or non-graceful. A detailed survey of graph labeling problems and the related results are shown in a survey by Gallian (2009). There is an unproved conjecture that all trees are graceful. Although, it is shown that trees with up to 27 vertices are graceful. It is shown that all cycles C_n are graceful if and only if $n \equiv 0$ or 3 (mod 4). All wheels W_n, Helms H_n, and Crowns R_n are graceful. The complete graphs K_n are graceful if and only if $n \leq 4$. The necessary condition for a windmill $K_n^{(m)}$ ($n \geq 3$) to be graceful is that $n \leq 5$; a windmill $K_n^{(m)}$ consists of m complete graphs K_n with one common vertex (Gallian, 2009). An example for each class of the graphs mentioned above and their graceful labelings are shown in Figure 2.

The graceful labeling problem is to find out whether a given graph is graceful or not, and if it is, how to label the vertices. The process of gracefully labeling a graph is a very tedious and difficult task for many classes of graphs (Eshghi & Azimi, 2004).

In the problem literature, many methods are presented for proving gracefulness of different classes of graphs theoretically, but most of them did not use a general method for finding the graceful labeling of the graphs to be studied. These theoretical methods focus on finding out whether the given graph is graceful or not, rather than finding its exact labeling. Therefore, the problem of finding a graceful labeling for a given graph, even when we know theoretically that the graph is graceful, is not yet targeted by many researchers. There exist only two mathematical programming methods for finding graceful labeling of graphs. The first one is a constraint programming approach (Redl, 2003), and the second one is a branch and bound (B&B) algorithm based on mathematical programming (Eshghi & Azimi, 2004). It is shown that the B&B method generates better solutions than the constraint programming method for graceful labeling problem (Eshghi & Azimi, 2004).

In this paper, a new general method for gracefully labeling any type of graphs is presented. Since this method inputs only the adjacency matrix of the graph, it is very easy to use; finally, it outputs the graceful labeling of the given graph, if any. In this approach, the graceful labeling problem is represented as an assignment-type problem with the aim of finding a feasible solution, and a metaheuristic approach based on Ant Colony Optimization (ACO) for gracefully labeling graphs is presented.

The rest of this paper is organized as follows: In the next section, Ant Colony Optimization is briefly introduced. Afterwards, a new representation of graceful labeling problem in a briefly modified framework of ACO, and the proposed metaheuristic algorithm are introduced. The final section shows the obtained results and compares them with those of the existing methods in the literature.