Chapter 18
Page Number and Graph Treewidth

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ABSTRACT

Book-embedding of graph G involves embedding its vertices along the spine of the book and assigning its edges to pages of the book such that no two edges cross on the same page. The pagenumber of G is the minimum number of pages in a book-embedding of G. In this paper, the authors also examine the treewidth TW(G), which is the minimum k such that G is a subgraph of a k-tree. The authors then study the relationship between pagenumber and treewidth. Results show that PN(G)≤TW(G), which proves a conjecture of Ganley and Heath showing that some known upper bounds for the pagenumber can be improved.

INTRODUCTION

The book-embedding problem for graph *G* is to embed its vertices onto a line along the spine of the book and to draw the edges on pages of book such that no two edges on the same page cross, and the number of used pages is minimized.

The book-embedding problem has been motivated by several areas of computer science such as VLSL theory, multilayer printed circuit boards (PCB), sorting with parallel stacks and Turning-machine and the design of fault-tolerant processor arrays, etc (e.g., Chung et al., 1987). The DIOGENES approach to fault-tolerant processor arrays, proposed by Rosenberg (1986), is the most famous one. In the DIOGENES approach, the processing elements are laid out in a logical line, and some number of bundles of wires run in parallel with the line. The faulty elements are bypassed, and the fault-free ones are interconnected through the bundles. Here, the bundles work as queues and/or stacks. If the bundles work as stacks, then the

realization of an interconnection network needs a book-embedding of the interconnection network. In this case, the number of pages corresponds to the number of bundles of the DIOGENES stack layout. Therefore, book-embeddings with few pages realize more hardware-efficient DIO-GENES stacks layouts.

The book-embedding problem can be stated as a graph-labeling problem as follows. We shall follow the graph-theoretic terminology and notation used by Bondy and Murty (1976) and Golumbic (1980).

Given a simple connected graph $G=(V,E)$ with n vertices, a bijection $f: V \rightarrow \{1, 2, \cdots, n\}$ is called a *labeling* of G by Chung(1988), where $f(v) \in \{1, 2, \cdots, n\}$ represents the label of vertex $v \in V$. Let $u_i = f^{-1}(i)$ be the vertex with label i. Then the labeling f can also be regarded as an ordering $(u_1, u_2, \cdots u_n)$ on a line. For a labeling f, two edges $uv, xy \in E$ are said to be *crossing* if
$$f(u) < f(x) < f(v) < f(y)$$
or $f(x) < f(u) < f(y) < f(v)$.

With respect to a labeling f, a partition $\mathcal{C} = (E_1, \cdots, E_p)$ of the edge set $E(G)$ is called a *page partition* if no two edges in any subset E_i $(1 \leq i \leq p)$ are crossing. This page partition can be thought of as a coloring of $E(G)$ where the edges in E_i have color i and no two edges of the same color are crossing. Thus, a page partition C represents an assignment of edges of G to pages of the book. We call the minimum number of subsets in a page partition C the page number of G under labeling f, and denote it by $PN(G,f)$. The pagenumber of G is then defined as

$$PN(G) = \min_f PN(G, f)$$

where f is taken over all labelings of G.

Even for a given labeling f, determining the pagenumber $PN(G,f)$ is known to be NP-complete (e.g., Chung et al., 1987). In other words, this is a hard problem in general. So, most researchers now are interested in the polynomially solvable special cases and lower or upper bounds of the pagenumber (e.g., Chung et al., 1987; Ganley & Heath, 2001). Recently, Ganley and Heath (2001) proved that $PN(G) \leq k+1$ if G is a k-tree. In the present paper we present a better result that $PN(G) \leq TW(G)$ for any graph G, where $TW(G)$ stands for the treewidth of G. This upper bound is sharp; and by means of it, we can improve the relations between pagenumber and other graph-theoretic parameters (such as pathwidth, bandwidth, cutwidth). In particular, the result on k-trees can be improved as $PN(G) \leq k$, and thus the conjecture of Ganley and Heath (2001) that every k-tree has a k-page embedding is proved to be true.

1. PRELIMINARIES

Let us recall some definitions and results on chordal graphs and k-trees.

A graph G is called a chordal graph if every cycle of length greater than three has a chord. Here, a chord means an edge joining two nonconsecutive vertices of the cycle. There have been many results on the characterization of chordal graphs in the literature (e.g., Blair & Peyton, 1993; Bodlaender, 1993; Golumbic, 1980; Kloks, 1994). The following is known as the clique-intersection property of chordal graphs, in which the tree T is called the *clique-tree* of G (e.g., Blair & Peyton, 1993; Golumbic, 1980).

Lemma 1. Let $\mathcal{X} = \{X_1, X_2, \cdots, X_m\}$ be the set of all maximal cliques of G. Then G is chordal if and only if there exists a tree T with vertex set $\mathcal{X}$ such that for each vertex v of G, the subgraph of T induced by $\{X \in \mathcal{X} \mid v \in X\}$ is connected and thus is a subtree of T.

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