Chapter XVI
Quaternionic Neural Networks: Fundamental Properties and Applications

Teijiro Isokawa
University of Hyogo, Japan

Nobuyuki Matsui
University of Hyogo, Japan

Haruhiko Nishimura
University of Hyogo, Japan

ABSTRACT

Quaternions are a class of hypercomplex number systems, a four-dimensional extension of imaginary numbers, which are extensively used in various fields such as modern physics and computer graphics. Although the number of applications of neural networks employing quaternions is comparatively less than that of complex-valued neural networks, it has been increasing recently. In this chapter, the authors describe two types of quaternionic neural network models. One type is a multilayer perceptron based on 3D geometrical affine transformations by quaternions. The operations that can be performed in this network are translation, dilatation, and spatial rotation in three-dimensional space. Several examples are provided in order to demonstrate the utility of this network. The other type is a Hopfield-type recurrent network whose parameters are directly encoded into quaternions. The stability of this network is demonstrated by proving that the energy decreases monotonically with respect to the change in neuron states. The fundamental properties of this network are presented through the network with three neurons.

INTRODUCTION

Complex numbers play an important role in practical applications and fundamental theorems in various fields of engineering such as electromagnetics, communication, control theory, and quantum mechanics. The application of complex numbers to neural networks has recently attracted attention because they tend to improve the learning ability and conform to the abovementioned applications (Hirose, 2003) (Rao, Nitta & Murthy, 2008).

They enable the modeling of a point in two-dimensional space as a single entity, rather than as a set of two data items on which 2D geometrical affine operations are performed. It has been shown that a neural network with the representation and operations of complex numbers results in improved performance of the geometrical affine transformation in two-dimensional space, whereas the performance of real-valued (conventional) neural networks is comparatively poor. The operations involving complex numbers would improve the performance of neural networks for processing two-dimensional data.

Let us consider the case in which the data are three dimensional, such as color images and body images. These data, of course, can be processed by many neurons of real- or complex-valued neural networks; however, the processing efficiency may be increased by implementing direct encoding in terms of hypercomplex numbers. Consequently, the application of hypercomplex numbers, particularly quaternions, to neural networks has been investigated. Quaternions are a class of hypercomplex number systems, a four-dimensional extension of imaginary numbers. One of benefits by quaternions is that affine transformation of geometric figures in three-dimensional space (3D geometrical affine transformation), especially spatial rotation, can be represented compactly and efficiently; in recent years, quaternions are extensively used in the fields of robotics, control of satellites, and computer graphics, etc.

In this chapter, we describe two models of neural networks based on quaternions and present their fundamental properties and possible applications. The first model is a multilayer perceptron based on the 3D geometrical affine transformations by quaternions. The operators in neurons adopt the 3D geometrical affine transformation. After the description of the neuron model and the error backpropagation algorithm for a training algorithm, four types of tasks, i.e., applications of three-bit parity check problem, affine transformation in three-dimensional space, color image compression, and color night vision, are introduced. The performances of the multilayer perceptron for these tasks are evaluated by comparing them with the performances of the real-valued networks for the same tasks. The other model is the Hopfield-type recurrent network where the states of neurons are directly encoded in quaternions. The quaternionic component of the neuron state takes a bipolar value (+1 or −1). The energy function for this network is defined, and it is proved that this function monotonically decreases with respect to the change in neuron states. The property of the stable points in the network is investigated by using an example of a network with three neurons.

BACKGROUND

Several systems for hypercomplex numbers have been investigated, such as quaternion, octonion, sedenion. They can be described as special cases of Clifford algebra (Porteous, 1995) (Lounesto, 2001). Quaternions are hypercomplex numbers of rank four; they are the four-dimensional extensions of imaginary numbers discovered by Sir William Rowan Hamilton (Graves, 1975) (Hankins, 1980) (Bell, 1999), which have been extensively used in modern mathematics, signal processing, computer graphics, robotics, etc (Lambek, 1995) (Mukundan, 2002) (Hoggar, 1992) (Kuipers, 1998) (Previn & Webb, 1983) (Sahul, Biswall & Subudhi, 2008) (Bülow & Sommer, 2001). Quaternion is also defined as a special case of Clifford algebra, but the representation of quaternion is simple and easy to understand its geometrical meanings. It has been found that 3D geometrical affine transformations—translation, dilatation, and spatial rotation—in three-dimensional space can be represented compactly and efficiently by the operators of quaternions. In the following section, we recapitulate the basic definitions, operations, and notations of quaternions. The 3D geometrical affine transformation realized by quaternions, which is used in the multilayer perceptron model in this chapter, is also described. For the detailed properties and applications of quaternions, please refer to the literatures (Lambek, 1995) (Mukundan, 2002) (Hoggar, 1992) (Kuipers, 1998).

Definition and Notation of Quaternion Algebra

Quaternions form a class of hypercomplex numbers that consist of a real number and three types of imaginary numbers: $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. Formally, a quaternion number is defined as a vector $\boldsymbol{x}$ in a four-dimensional vector space,

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